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Triply periodic constant mean curvature surfaces

William H. Meeks III* Giuseppe Tinaglia†

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Abstract

Given a closed flat 3-torus N , for each $H > 0$ and each non-negative integer g , we obtain area estimates for closed surfaces with genus g and constant mean curvature H embedded in N . This result contrasts with the theorem of Traizet [31], who proved that every flat 3-torus admits for every positive integer g with $g \neq 2$, connected closed embedded minimal surfaces of genus g with arbitrarily large area.

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1 Introduction

In [31] Traizet proved that every flat 3-torus admits for every positive integer g with $g \neq 2$, connected closed embedded minimal surfaces of genus g with arbitrarily large area. In contrast to Traizet's result we prove the following area estimates for closed embedded surfaces of constant positive mean curvature in a flat 3-torus.

Theorem 1.1 *Given $a, b, d, I_0 \in (0, \infty)$ with $a \leq b$ and $g \in \mathbb{N} \cup \{0\}$, there exists $A(g, a, b, d, I_0) > 0$ such that the following hold. Let N be a flat 3-torus with an upper bound d on its diameter and a lower bound I_0 for its injectivity radius and let M be a possibly disconnected, closed surface embedded in N of genus g with constant mean curvature $H \in [a, b]$. Then:*

$$\text{Area}(M) \leq A(g, a, b, d, I_0).$$

Interest in results like the above area estimates arises in part from the fact that triply periodic surfaces of constant mean curvature in \mathbb{R}^3 , which are always the lifts of surfaces of constant mean curvature in a flat 3-torus, occur in nature. For example,

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approximations to these special surfaces appear in studies of Fermi surfaces (equipotential surfaces) in solid state physics and in the geometry of liquid crystals. They are also found in material sciences where they closely approximate surface interfaces in certain inhomogeneous mixtures of two different compounds and as the boundary of the microscopic calcium deposit patterns in sea urchin shells. The geometry of triply-periodic constant mean curvature surfaces arising in nature have profound consequences for the physical properties of the materials in which they occur, which is one of the reasons why they are studied in great detail by physical scientists. We refer the interested reader to the science articles [1, 9, 28] for further readings along these lines.

For the purpose of exposition, in this paper we will call an orientable immersed surface M in an oriented Riemannian 3-manifold an H -surface if M is *embedded* and it has *non-zero constant mean curvature* H , where we will assume that H is positive by appropriately orienting M .

A consequence of Theorem 1.1 and its proof is that for $H > 0$ fixed, the moduli space of closed connected H -surfaces of fixed genus in a flat 3-torus has a natural compactification as a compact real semi-analytic variety in much the same way that one can compactify the moduli space of closed connected Riemann surfaces of fixed genus. In this regard it is worthwhile to compare our results with the area estimates of Choi and Wang [4] for closed embedded minimal surfaces of fixed genus in the 3-sphere $\langle \mathbb{S}^3, h \rangle$ with a metric h of positive Ricci curvature, and with the result by Choi and Schoen [3] that the moduli space of fixed genus closed minimal surfaces embedded in $\langle \mathbb{S}^3, h \rangle$ has the structure of a compact real analytic variety.

Theorem 1.2 *Let N be a flat 3-torus. Given $a, b \in (0, \infty)$ with $a \leq b$ and $g \in \mathbb{N} \cup \{0\}$, let $\{M_n\}_{n \in \mathbb{N}}$ be a sequence of closed H_n -surfaces in N , $H_n \in [a, b]$, of genus g . Then there exist a subsequence of $\{M_n\}_{n \in \mathbb{N}}$ and a non-empty, possibly disconnected, strongly Alexandrov embedded¹ surface M_∞ of constant mean curvature $H_\infty \in [a, b]$ and of genus at most g , such that the following holds. The surfaces in this subsequence converge smoothly with multiplicity one to M_∞ away from a finite set of points Δ (of singular points of convergence) contained in the self-intersection set of M_∞ .*

Suppose that $\{M_n\}_{n \in \mathbb{N}}$ is the subsequence that converges to the limit surface M_∞ in the above theorem with singular set of convergence Δ . The *convergence with multiplicity one* property implies that the areas of M_n converge to the area of the limit surface. In a sense that is made clear in Proposition 9.2, nearby every $q \in \Delta$ (on the scale of the injectivity radius), the surface M_n contains domains that look like a scaled catenoid for n large. It also follows from Proposition 9.2 that M_∞ has area density 2 at each $q \in \Delta$. We conjecture that for every $q \in \Delta$ there exists $\varepsilon > 0$ small such that $B_N(q, \varepsilon) \cap M_n$ is an annulus for n sufficiently large, where $B_N(q, \varepsilon)$ denotes the open ball of N centered at q of radius ε (see Conjecture 12.1).

¹A closed immersed surface $f: \Sigma \rightarrow N$ of positive mean curvature in N is called *strongly Alexandrov embedded* if f extends on the mean convex side of its image to an immersion of a compact 3-manifold W with $\Sigma = \partial W$, where the extended immersion is injective on the interior of W . In particular, such a strongly Alexandrov embedded Σ can be approximated by embedded surfaces on its mean convex side.

The manuscript is organized as follows. In Section 2 we list a few results from other papers that will be needed in the proof of Theorem 1.1. In Section 3 we describe some separation properties of certain H -surfaces in a complete flat 3-manifold. In Sections 4, 5, 6, 7 and 8 we prove a version of Theorem 1.1, where the area bound depends in a non-specified manner on the ambient 3-torus N . Arguing by contradiction, we suppose there exists a sequence $\{M_n\}_{n \in \mathbb{N}}$ of closed H_n -surfaces, $H_n \in [a, b]$, of genus g with arbitrary large area in N . Then, in Section 4 we prove that, after replacing by a subsequence, the injectivity radii of the surfaces M_n must be going to zero. In Section 5 we analyze the local geometry of M_n nearby a point where the injectivity radius is becoming arbitrarily small, such points are called *singular points*. In Section 6 we analyze a global consequence for the geometry of M_n if there exist singular points. In Section 7 we use this analysis to prove that the number of singular points is finite. In Section 8 we apply the fact that the number of singular points is finite to obtain a contradiction to the assumed hypothesis that the area is becoming arbitrarily large. In Section 9 we prove the compactness theorem, namely Theorem 1.2, as well as a stronger characterization of the geometry of M_n in a small neighborhood of singular point, when n is sufficiently large. In Section 10 we analyze the constant $A(g, a, b, d, I_0)$ that gives the area bound in Theorem 1.1 and complete its proof. In Section 11 we discuss the dependence of $A(g, a, b, d, I_0)$ on the genus g , on the lower and upper bounds a and b for the constant mean curvature H and on the diameter d of the flat 3-torus, under the assumption that I_0 is fixed. In Section 12, we promote several conjectures related to our main theorems.

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2 Preliminaries

We now recall several key notions and theorems from [17] that we will apply in the next section.

Definition 2.1 Let M be an H -surface, possibly with boundary, in a complete oriented Riemannian 3-manifold N .

1. M is an H -disk if M is diffeomorphic to a closed disk in the complex plane.
2. $|A_M|$ denotes the *norm of the second fundamental form* of M .
3. The *radius* of a Riemannian n -manifold with boundary is the supremum of the intrinsic distances of points in the manifold to its boundary.

The next two results are contained in [17], see also [22, 16, 18, 21, 19].

Theorem 2.2 (Radius Estimates) *There exists an $\mathcal{R} \geq \pi$ such that any H -disk in \mathbb{R}^3 has radius less than \mathcal{R}/H .*

Theorem 2.3 (Curvature Estimates) *Given $\delta, \mathcal{H} > 0$, there exists a $K(\delta, \mathcal{H}) \geq \sqrt{2}\mathcal{H}$ such that for any H -disk $\mathcal{D} \subset \mathbb{R}^3$ with $H \geq \mathcal{H}$,*

$$\sup_{\{p \in \mathcal{D} \mid d_{\mathcal{D}}(p, \partial \mathcal{D}) \geq \delta\}} |A_{\mathcal{D}}| \leq K(\delta, \mathcal{H}).$$

The following notions of injectivity radius function and injectivity radius are needed in the statement and the proof of Theorem 1.1.

Definition 2.4 The injectivity radius $I_M(p)$ at a point p of a complete Riemannian manifold M is the supremum of the radii R for which the exponential map on the open ball of radius R in $T_p N$ is a diffeomorphism. This defines the *injectivity radius function*, $I_M: M \rightarrow (0, \infty]$, which is continuous on M (see e.g., Proposition 88 in Berger [2]). The infimum of I_M is called the *injectivity radius* of M .

Corollary 2.5 *If M is a complete H -surface in \mathbb{R}^3 with positive injectivity radius r_0 , then*

$$\sup_M |A_M| \leq K(r_0, H).$$

Furthermore, such an M is properly embedded in \mathbb{R}^3 and it is the oriented boundary of a smooth, possibly disconnected, mean convex closed domain G_M in \mathbb{R}^3 which has a 1-sided regular ε -neighborhood for its boundary for some $\varepsilon > 0$.

Proof. The first statement in the corollary is an immediate consequence of Theorem 2.3. We next explain how the second statement follows from the first one. First note that complete connected H -surfaces in \mathbb{R}^3 of bounded norm of the second fundamental form are proper; see for instance [13] for this result. It follows from [13] that for some $\varepsilon > 0$, each component of M has a regular ε -neighborhood on its mean convex side, where $\varepsilon > 0$ only depends on the bound of the norm of the second fundamental form of M . In [25] Ros and Rosenberg proved that given two complete disjoint connected proper H -surfaces in \mathbb{R}^3 , neither one lies on the mean convex side of the other one. It then follows from elementary separation properties that a complete, possibly disconnected, H -surface M in \mathbb{R}^3 of bounded norm of the second fundamental form is proper, it is the oriented boundary of a possibly disconnected, mean convex closed domain G_M in \mathbb{R}^3 and, for some $\varepsilon > 0$, G_M has a 1-sided regular ε -neighborhood for its boundary M . \square

For a point p in a Riemannian manifold N , we let $\mathbb{B}_N(p, R)$ denote the open ball of N centered at p of radius R and when $N = \mathbb{R}^3$, we let $\mathbb{B}(R)$ be the ball centered at the origin of radius R ; we let $\overline{\mathbb{B}}_N(p, R)$ denote the related closed balls.

Next we state a result that is closely related to Corollary 2.5 and that is needed in the proof of Theorem 1.1. By applying the techniques used to prove Theorem 3.5 in [20], one obtains the following result.

Proposition 2.6 *Given $R > 0$, $\alpha > 0$ and $\beta > 0$, there exists a constant $\omega(R, \alpha, \beta)$ such that the following holds. Suppose $M \subset \overline{\mathbb{B}}(R)$ is an H -surface with $\partial M \subset \partial \mathbb{B}(R)$,*

$H \geq \alpha$ and $\sup_M |A_M| < \beta$ and such that M bounds a mean convex domain in $\mathbb{B}(R)$. Then

$$\text{Area}(M \cap \mathbb{B}(R/2)) < \omega(R, \alpha, \beta).$$

We next describe the notion of flux of an H -surface in \mathbb{R}^3 , see for instance [5, 6, 30] for further discussion of this invariant.

Definition 2.7 Let γ be a piecewise-smooth 1-cycle in an H -surface M in \mathbb{R}^3 . The *flux* of γ is $F(\gamma) = |\int_\gamma (H\gamma + \xi) \times \dot{\gamma}|$, where ξ is the unit normal to M along γ and γ is parameterized by arc length. The flux only depends on the homology class of γ .

In the case that $H_1(M) = \mathbb{Z}$, we let $F(M)$ denote the flux of any curve which represents a generator of $H_1(M)$.

The next theorem appears in [17].

Theorem 2.8 *Given $\rho > 0$ and $\delta \in (0, 1)$ there exists a positive constant $I_0(\rho, \delta)$ such that if E is a compact 1-annulus with $F(E) \geq \rho$ or with $F(E) = 0$, then*

$$\inf_{\{p \in E \mid d_E(p, \partial E) \geq \delta\}} I_E \geq I_0(\rho, \delta),$$

where $I_E: E \rightarrow [0, \infty)$ is the injectivity radius function of E .

Corollary 2.5 and Theorem 2.8 imply that annular ends of complete H -surfaces in \mathbb{R}^3 have representatives which are properly embedded in \mathbb{R}^3 , and so by the classification results in [5], we have the following.

Corollary 2.9 *Annular ends of complete H -surfaces in \mathbb{R}^3 have representatives that are asymptotic to the ends of unduloids.*

3 Properties of H -surfaces in a flat 3-manifold

In this section we prove some geometric properties of complete H -surfaces in a flat 3-manifold.

Theorem 3.1 *Let N be a complete connected flat 3-manifold with universal cover $\Pi: \mathbb{R}^3 \rightarrow N$ and let M be a complete H -surface in N . Then the following holds:*

1. *If M has positive injectivity radius, then it has bounded norm of the second fundamental form, is properly embedded in N and it is the oriented boundary of a smooth, possibly disconnected, complete closed subdomain G_M on its mean convex side. The mean convex domain G_M has radius at most $1/H$, and $M = \partial G_M$ has a 1-sided regular ε -neighborhood in G_M for some $\varepsilon > 0$.*
2. *If M has finite topology, then it has positive injectivity radius. Furthermore, each annular end E of M lifts to an annulus $\tilde{E} \subset \mathbb{R}^3$, where \tilde{E} is asymptotic to the end of an unduloid.*

Proof. We first prove item 1 of the theorem. Suppose that M has positive injectivity radius. Consider the possibly disconnected surface $\Sigma = \Pi^{-1}(M) \subset \mathbb{R}^3$, which also has positive injectivity radius. Applying Corollary 2.5 to Σ , we conclude that Σ is properly embedded in \mathbb{R}^3 with bounded norm of the second fundamental form and Σ is the boundary of a mean convex closed domain G_Σ that has a regular ε -neighborhood for its boundary surface Σ for some $\varepsilon > 0$. By elementary theory of covering spaces, it follows that the domain $G_M = \Pi(G_\Sigma)$ satisfies all of the properties in item 1 of the theorem except possibly for the property that the radius of G_M is at most $1/H$. Arguing by contradiction suppose that there exists a point $p \in G_M$ with $d_N(p, \partial G_M) = R_0 > 1/H$, where d_N is the distance function in N . In this case consider the associated mean convex region $G_\Sigma = \Pi^{-1}(G_M) \subset \mathbb{R}^3$. For any choice $\tilde{p} \in \Pi^{-1}(p) \cap G_\Sigma$, $d_{\mathbb{R}^3}(\tilde{p}, \partial G_\Sigma) = R_0 > 1/H$, where $d_{\mathbb{R}^3}$ is the Euclidean distance. Note that $\partial \mathbb{B}(\tilde{p}, R_0)$ intersects ∂G_Σ at some point q and $\partial \mathbb{B}(\tilde{p}, R_0) \subset G_\Sigma$. Since the mean curvature of ∂G_Σ is H and the mean curvature of $\partial \mathbb{B}(\tilde{p}, R_0)$ is $1/R_0 < H$, we obtain a contradiction to the mean curvature comparison principle applied to the surfaces $\partial \mathbb{B}(\tilde{p}, R_0)$ and ∂G_Σ at the point q . This completes the proof of item 1 of Theorem 1.1.

We next prove item 2. Let E be an annular end representative of M . We claim that the inclusion map $i: E \rightarrow M \subset N$ lifts through $\Pi: \mathbb{R}^3 \rightarrow N$ to $\tilde{i}: E \rightarrow \mathbb{R}^3$, from which item 2 follows by applying Corollary 2.9 to $\tilde{i}(E)$. Clearly each component of $\Pi^{-1}(E) \subset \mathbb{R}^3$ must be an annulus A with boundary $\Pi(\partial A) = \partial E$. Otherwise, elementary covering space theory implies that each component of $\Pi^{-1}(E) \subset \mathbb{R}^3$ is a complete simply connected H -surface with boundary and having infinite radius. Such a component contains H -disks of arbitrarily large radius, thereby contradicting Theorem 2.2. Suppose that the inclusion map $i: E \rightarrow M \subset N$ does not lift through $\Pi: \mathbb{R}^3 \rightarrow N$ to a continuous map $\tilde{i}: E \rightarrow \mathbb{R}^3$. In this case there exist $p, q \in A$, $p \neq q$ such that $\Pi(p) = \Pi(q) \in i(E)$. Let $\sigma: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the covering transformation such that $\sigma(p) = q$. The map σ is an isometry of \mathbb{R}^3 that leaves invariant the compact set ∂A . This implies that σ has a fixed point and therefore cannot be a nontrivial covering transformation. This contradiction completes the proof of item 2, thus finishing the proof of the theorem. \square

4 Area estimates for H -surfaces in a flat 3-torus

In the next six sections, N will denote a flat 3-torus. We begin by proving a weaker area estimate than the one in Theorem 1.1, namely an area estimate for M that depends on the flat 3-torus but without specifying what geometric quantities of the 3-torus are important in such estimates. In Section 10 we outline how Theorem 1.1 will follow from Theorem 4.1 below. Notice that the H -surface M in the next theorem is assumed to be *connected*, whereas the surface in Theorem 1.1 may be disconnected. Indeed, the next theorem actually holds in the more general situation where M is allowed to be disconnected. This follows from the fact that the number of components of M can be bounded in terms of its genus and the geometry of N ; see Section 8.1.

Theorem 4.1 *Let N be a flat 3-torus. Given $a, b \in (0, \infty)$, with $a \leq b$, and $g \in \mathbb{N} \cup \{0\}$, there exists a constant $A(N, a, b, g)$ such that if M is a closed connected H -surface in N of genus g with $H \in [a, b]$, then*

$$\text{Area}(M) \leq A(N, a, b, g).$$

Proof. Arguing by contradiction, suppose M_n is a sequence of H_n -surfaces, $H_n \in [a, b]$, in N of genus g such that

$$\text{Area}(M_n) > n.$$

The proof of Theorem 4.1 is divided into steps with the final contradiction appearing at the end of Section 8.

Note that by item 1 of Theorem 3.1, M_n separates N into two regions and one of them, denoted by G_{M_n} , is mean convex.

Claim 4.2 *The injectivity radii $I(M_n)$ converge to zero as n goes to infinity.*

Proof. Arguing by contradiction, suppose there exists $\delta > 0$ such that after replacing by a subsequence, $I(M_n) > \delta$ for any n . Then by Theorem 2.3, the set of functions $\{|A_{M_n}|\}_n$ is bounded from above by a fixed constant independent of n . Since the surfaces M_n are H_n -surfaces with $H_n \geq a > 0$, then Theorem 3.5 in [20] implies that there exist constants $\varepsilon, A_0 > 0$ such that the surfaces have a 1-sided regular ε -neighborhood $\mathcal{N}(n, \varepsilon) \subset G_{M_n}$ and the area of each M_n is at most $A_0 \cdot \text{Volume}(\mathcal{N}(n, \varepsilon))$. Therefore,

$$\text{Area}(M_n) \leq A_0 \text{Volume}(\mathcal{N}(n, \varepsilon)) \leq A_0 \text{Volume}(N),$$

which contradicts that the areas of the surfaces M_n are becoming arbitrarily large. \square

In light of Claim 4.2 we introduce the following definitions.

Definition 4.3 Let U be an open set in N .

1. We say that a sequence of surfaces $T_n \subset U$ has *locally bounded norm of the second fundamental form in U* if for each compact ball B in U , the norms of the second fundamental forms of the surfaces $T_n \cap B$ are uniformly bounded.
2. We say that a sequence of surfaces $T_n \subset U$ has *locally positive injectivity radius in U* if for each compact ball B in U the injectivity radius functions of the surfaces T_n are bounded away from 0 in $T_n \cap U$.

Suppose that the injectivity radius functions I_n of M_n have their minimum values at points $p_{1,n} \in M_n$. By Claim 4.2 we can assume that $I_n(p_{1,n}) < 1/n$. After choosing a subsequence and reindexing, we obtain a sequence $M_{1,n}$ such that the points $p_{1,n} \in M_{1,n}$ converge to a point $q_1 \in N$. Suppose the sequence of surfaces $M_{1,n}$ fails to have locally bounded injectivity radius in $N - \{q_1\}$. Let $q_2 \in N - \{q_1\}$ be a point that is furthest away from q_1 and such that, after passing to a subsequence $M_{2,n}$, there exists a sequence of points $p_{2,n} \in M_{2,n}$ converging to q_2 with $\lim_{n \rightarrow \infty} I_n(p_{2,n}) = 0$. If the sequence of

surfaces $M_{2,n}$ fails to have locally bounded injectivity radius in $N - \{q_1, q_2\}$, then let $q_3 \in N - \{q_1\} \cup \{q_2\}$ be a point in N that is furthest away from $\{q_1, q_2\}$ and such that, after passing to a subsequence, there exists a sequence of points $p_{3,n} \in M_{3,n}$ converging to q_3 with $\lim_{n \rightarrow \infty} I_n(p_{3,n}) = 0$.

Continuing inductively in this manner and using a diagonal-type argument, we obtain after reindexing, a new subsequence M_n (denoted in the same way) and a countable (possibly finite) non-empty set $\Delta' := \{q_1, q_2, q_3, \dots\} \subset \mathbf{N}$ such that for every $k \in \mathbb{N}$, there is an integer $N(k)$ such that for $n \geq N(k)$, there exist points $p(n, q_k) \in M_n \cap B_N(q_k, 1/n)$ where $I_{M_n}(p(n, q_k)) < 1/n$. We let Δ denote the closure of Δ' in N . It follows from the construction of Δ that the sequence M_n has locally positive injectivity radius in $N - \Delta$.

We call the set Δ the *singular set of convergence* and $q \in \Delta$ a *singular point*. Note that, by using Theorem 2.3, M_n has locally positive injectivity radius in $N - \Delta$ if and only if M_n has locally bounded norm of the second fundamental form in $N - \Delta$. In later sections we will replace the sequence M_n by some subsequence; a key property that follows from our construction of M_n is that Δ continues to be the singular set of convergence for this new subsequence.

5 The local geometry around singular points

In this section we study the geometry of M_n nearby points in Δ . Let $q \in \Delta$. Then by definition and after possibly replacing the surfaces M_n by a subsequence, there exists a sequence of points $p_n \in M_n$ such that $d_N(p_n, q) < 1/n$, where d_N is the distance function in N , and

$$\frac{1}{nI_n(p_n)} > n.$$

Consider the continuous functions $h_n: M_n \cap \bar{B}_N(p_n, 1/n) \rightarrow \mathbb{R}$ given by

$$h_n(x) = \frac{d_N(x, \partial B_N(p_n, 1/n))}{I_{M_n}(x)}.$$

As h_n vanishes on $M_n \cap \partial B_N(p_n, 1/n)$ then there exists a point $p'_n \in M_n \cap B_N(p_n, 1/n)$ that is a point where h_n takes on its maximum value. This point is said to be a point of *almost-minimal injectivity radius*.

Let $\sigma_n := d_N(p'_n, \partial B_N(p_n, 1/n))$. Then

$$\frac{\sigma_n}{I_{M_n}(p'_n)} = h_n(p'_n) \geq h_n(p_n) = \frac{1}{nI_{M_n}(p_n)} > n. \quad (1)$$

Let M'_n be the compact surface $M_n \cap \bar{B}_N(p'_n, \sigma_n/2)$. Note that σ_n goes to zero as n goes to infinity and since M_n is compact with constant mean curvature $H \leq b$, then for n sufficiently large, M'_n is a compact surface with non-empty boundary that is contained in $\partial B_N(p'_n, \sigma_n/2)$. Moreover, given $q \in M'_n$ then

$$\frac{\sigma_n/2}{I_{M_n}(q)} \leq h_n(q) \leq h_n(p'_n) = \frac{\sigma_n}{I_{M_n}(p'_n)},$$

which implies that

$$I_{M_n}(q) \geq \frac{I_{M_n}(p'_n)}{2}. \quad (2)$$

Let I_0 denote the injectivity radius of N . Thinking about the balls in N with exponential coordinates, we will consider $\bar{B}_N(p'_n, r)$, $r \leq I_0$, to be the closed ball $\bar{\mathbb{B}}(r) = B_{\mathbb{R}^3}(\vec{0}, r) \subset \mathbb{R}^3$ of radius r centered at the origin. After defining $\lambda_n = \frac{1}{I_{M_n}(p'_n)}$, let

$$\widetilde{M}_n := \lambda_n M'_n \subset \bar{\mathbb{B}}(\lambda_n \frac{\sigma_n}{2}) \subset \mathbb{R}^3, \quad \partial \widetilde{M}_n \subset \partial \bar{\mathbb{B}}(\lambda_n \frac{\sigma_n}{2}).$$

Note that $I_{\widetilde{M}_n}(\vec{0}) = 1$. Since $H_n \leq b$, the constant mean curvature of \widetilde{M}_n goes to zero as n goes to infinity. By equation (1), it follows that $\lambda_n \frac{\sigma_n}{2}$ goes to infinity as n goes to infinity. By equation (2), the sequence of surfaces \widetilde{M}_n has locally positive injectivity radius in \mathbb{R}^3 ; indeed given B a compact set of \mathbb{R}^3 , it follows that for n sufficiently large, for any $q \in \widetilde{M}_n \cap B$ we have $I_{\widetilde{M}_n}(q) \geq 1/2$. However, since the constant mean curvature of \widetilde{M}_n is going to zero as n goes to infinity, it is no longer true for \widetilde{M}_n that having locally positive injectivity radius in \mathbb{R}^3 is equivalent to having locally bounded norm of the second fundamental form in \mathbb{R}^3 .

There are two cases to consider. Either \widetilde{M}_n has locally bounded norm of the second fundamental form in \mathbb{R}^3 or not. If \widetilde{M}_n does NOT have locally bounded norm of the second fundamental form in \mathbb{R}^3 then by Theorem 1.5 in [21], the surfaces \widetilde{M}_n converge on compact subsets of \mathbb{R}^3 to a minimal parking garage structure of \mathbb{R}^3 with two oppositely oriented columns, see Figure 1 and see for instance [10] for a detailed description of this limit object.

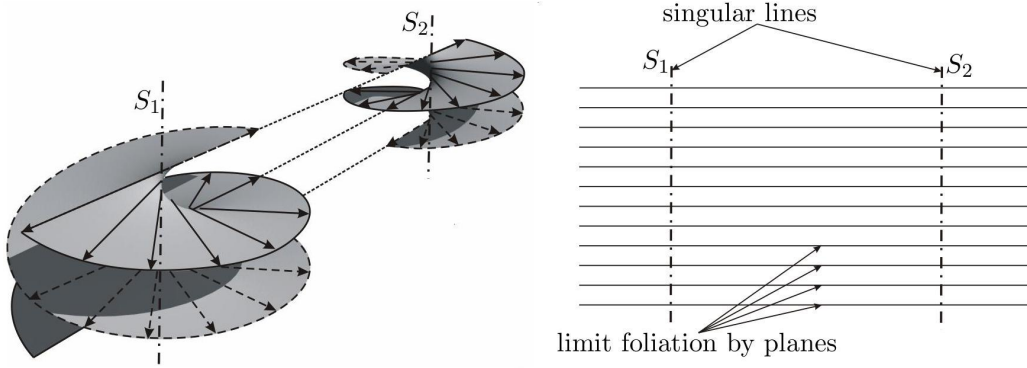


Figure 1: Parking garage structure: in this picture, the sequence of surfaces on the left hand side converge smoothly away from the union $S_1 \cup S_2$ or two straight lines orthogonal to the foliation of horizontal planes described on the right hand side.

If \widetilde{M}_n has locally bounded norm of the second fundamental form in \mathbb{R}^3 then by applying Theorem 1.3 in [21] and after replacing by a subsequence, the surfaces \widetilde{M}_n converge with multiplicity one or two on compact subsets of \mathbb{R}^3 to a properly embedded

minimal surface \widetilde{M}_∞ , in the sense that any sufficiently small normal neighborhood of any smooth compact domain Ω of the limit surface must intersect \widetilde{M}_n in one or two components that are small normal graphs over Ω for n large. Moreover \widetilde{M}_∞ has bounded norm of the second fundamental form, genus at most g and its injectivity radius at the origin is one. The surface \widetilde{M}_∞ either has genus zero or positive genus. If \widetilde{M}_∞ has genus zero and one end, then \widetilde{M}_∞ is simply-connected. Therefore, by the Cartan-Hadamard Theorem, it would have infinite injectivity radius at each point; this cannot happen because the injectivity radius of \widetilde{M}_∞ at the origin is one. Therefore if \widetilde{M}_∞ has genus zero, then it must have more than one end and there are two cases: \widetilde{M}_∞ is either a catenoid (finite topology [8]), or \widetilde{M}_∞ is a Riemann minimal example (infinite topology [11]); see Figure 2 and see for instance [11] for a detailed description of a Riemann minimal example.

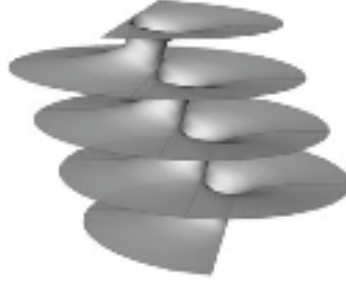


Figure 2: Riemann minimal example.

In summary, in the limit we obtain one of the examples listed below:

1. a catenoid or a Riemann minimal example;
2. a minimal parking garage structure with two oppositely oriented columns;
3. a properly embedded minimal surface with positive genus at most g .

Proposition 5.1 *The surfaces \widetilde{M}_n converge with multiplicity one or two on compact subsets of \mathbb{R}^3 to a catenoid or to a properly embedded minimal surface with positive genus at most g .*

Proof. In light of the previous discussion, we need to rule out the occurrence of a limit surface that is a Riemann minimal example or a minimal parking garage structure of \mathbb{R}^3 with two oppositely oriented columns. We next rule out the case that a Riemann minimal example occurs. An analogous discussion rules out the possibility that a minimal parking garage structure with two oppositely oriented columns occurs; see Remark 5.2.

Suppose that the limit object is a Riemann minimal example which we denote by \mathcal{R} . The surface \mathcal{R} is a properly embedded minimal planar domain in \mathbb{R}^3 of infinite topology which is foliated by circles and lines in a family of parallel planes. Given $R > 0$ let $\Omega(R) := \mathbb{B}(R) - \mathcal{R}$. Then, given $m \in \mathbb{N}$, there exists $R > 0$ such that $\mathbb{B}(R) \cap \mathcal{R}$ has at least $2m$ boundary components in $\partial\mathbb{B}(R)$ such that the following holds. At least m of these boundary components do not bound a disk whose interior is contained in $\Omega(R)$ and each pair of curves in this collection of boundary components does not bound an annulus whose interior is contained in $\Omega(R)$, see Figure 3. Thus, by the nature of

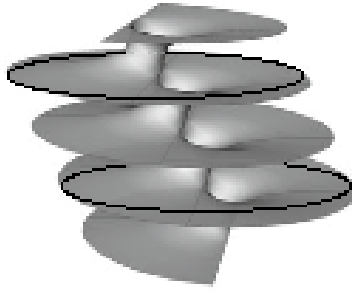


Figure 3: These curves are homotopically non-trivial and do not bound an annulus.

the convergence, a geometric picture that is similar to the one just described for \mathcal{R} is true for \widetilde{M}_n . Namely, let $\widetilde{M}_n(R)$ denote the connected component of $\widetilde{M}_n \cap \mathbb{B}(R)$ that contains the origin. Given $m \in \mathbb{N}$ there exists $R > 0$ such that for n sufficiently large $\partial\widetilde{M}_n(R)$ contains at least m simple closed curves

$$\{\Gamma_1(n), \dots, \Gamma_m(n)\} \quad (3)$$

and the following holds.

- Each $\Gamma_i(n)$, $i := 1, \dots, m$ does not bound a disk whose interior is contained in $\mathbb{B}(R) - \widetilde{M}_n(R)$.
- Each pair of curves $\{\Gamma_i(n), \Gamma_j(n)\}$, $i \neq j$, does not bound an annulus whose interior is contained in $\mathbb{B}(R) - \widetilde{M}_n(R)$.

Remark 5.2 The same description holds when the case of a minimal parking garage structure with two oppositely oriented columns occurs. In this case, the simple closed curves $\Gamma_i(n)$, $i := 1, \dots, m$, wind once around the pair of columns $S_1 \cup S_2$ in Figure 1.

Let $\{\gamma_1(n), \dots, \gamma_m(n)\}$ be the collection of curves in M_n corresponding to the curves $\{\Gamma_1(n), \dots, \Gamma_m(n)\}$ in $\partial\widetilde{M}_n(R)$.

Claim 5.3 *For n sufficiently large, the curve $\gamma_i(n)$, $i = 1, \dots, m$, is NOT homotopically trivial in M_n and each pair of curves $\{\gamma_i(n), \gamma_j(n)\}$, $i \neq j$, does NOT bound an annulus in M_n .*

Proof. Recall that the origin in \mathbb{R}^3 corresponds to points $p'_n \in M_n$ and let $B(\varepsilon)$ denote the balls of radius ε in N centered at p'_n . Recall that by item 1 of Theorem 3.1, M_n separates N into two closed regions and one of them, denoted by G_{M_n} is mean convex. By the previous discussion there exists $\varepsilon_n > 0$ such that $\gamma_i(n) \subset \partial B(\varepsilon_n)$, $\gamma_i(n)$ is homotopically non-trivial in $\overline{B}(\varepsilon_n) \cap G_{M_n}$ and each pair of curves $\{\gamma_i(n), \gamma_j(n)\}$, $i \neq j$, does not bound an annulus in $\overline{B}(\varepsilon_n) \cap G_{M_n}$. Moreover, $\lim_{n \rightarrow \infty} \varepsilon_n = 0$.

Suppose that $\gamma_i(n)$ is homotopically trivial in M_n , namely that $\gamma_i(n)$ bounds a disk in M_n . Then, by the results in [24], solving the annular Plateau problem for $\gamma_i(n)$ in G_{M_n} produces an embedded least-area disk $D_n \subset G_{M_n}$ with $\partial D_n = \gamma_i(n) \subset \partial B(\varepsilon_n)$. By construction, ∂D_n is homotopically non-trivial in $B(\varepsilon_n) \cap G_{M_n}$ and thus D_n cannot be contained in $B(\varepsilon_n) \cap G_{M_n}$. Hence, the disk D_n lifts to a minimal disk in \mathbb{R}^3 that is not contained in $\mathbb{B}(\varepsilon_n)$ but with boundary contained in $\partial \mathbb{B}(\varepsilon_n)$. This violates the convex hull property for minimal surfaces and gives a contradiction, which proves that $\gamma_i(n)$ is homotopically non-trivial in M_n .

Suppose that the pair of curves $\{\gamma_i(n), \gamma_j(n)\}$, $i \neq j$, bounds an annulus in M_n . Since these curves are homotopically non-trivial, solving the annular Plateau's problem in G_{M_n} (see Dehn's Lemma for Planar Domains in [23] as adapted by the more general boundary conditions in [24]) for $\{\gamma_i(n), \gamma_j(n)\}$, $i \neq j$ yields an embedded least-area annulus $A_n \subset G_{M_n}$ with $\partial A_n = \gamma_i(n) \cup \gamma_j(n) \subset \partial B(\varepsilon_n)$. By construction, A_n is not contained in $B(\varepsilon_n) \cap G_{M_n}$. Let $\mathbb{B}(z_k, \varepsilon_n)$ denote a lift of $B(\varepsilon_n)$ into \mathbb{R}^3 and note that if $k_1 \neq k_2$ then $d_{\mathbb{R}^3}(z_{k_1}, z_{k_2}) \geq I_0$. Since $\varepsilon_n < I_0$ and $\gamma_i(n), \gamma_j(n) \subset \partial B(\varepsilon_n)$, each closed curve lifts to a closed curve in $\partial \mathbb{B}(z_k, \varepsilon_n)$. Abusing the notation, let $\gamma_i(n)$ denote the lift of $\gamma_i(n)$ in \mathbb{R}^3 that is contained in $\mathbb{B}(z_1, \varepsilon_n)$. Let \tilde{A}_n be the annulus that is the lift of A_n that has $\gamma_i(n)$ as one of its boundary component; note that \tilde{A}_n is not contained in $\mathbb{B}(z_1, \varepsilon_n)$. Suppose that the other boundary component of \tilde{A}_n is contained in a certain $\mathbb{B}(z_k, \varepsilon_n)$ where $z_k \neq z_1$. This being the case, the boundary of \tilde{A}_n consists of two components $\gamma_i(n)$ and $\gamma_j(n)$ with $\gamma_i(n) \subset \mathbb{B}(z_1, \varepsilon_n)$, $\gamma_j(n) \subset \mathbb{B}(z_k, \varepsilon_n)$, $d_{\mathbb{R}^3}(z_1, z_k) \geq I_0$ and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. This violates a well-known property of minimal surfaces (pass a catenoid between the two balls until a first point of interior contact). This shows that the other boundary component of \tilde{A}_n must also be contained in $\mathbb{B}(z_1, \varepsilon_n)$. This being the case, the fact that \tilde{A}_n is not contained in $\mathbb{B}(z_1, \varepsilon_n)$ violates the convex hull property for minimal surfaces. Thus, we have obtained a contradiction and completed the proof of the claim. \square

In order to finish the proof of the proposition, it suffices to show that if a Riemann minimal example occurs, then there are two curves $\gamma'_1(n)$ and $\gamma'_2(n)$ as described in the previous claim and which bound an annulus in M_n . This will be a simple consequence of the following lemma.

Lemma 5.4 *Let Σ be a closed, possibly disconnected, surface of positive genus g and let Γ be a collection of simple closed curves in Σ that are homotopically non-trivial and*

pair-wise disjoint. If the number of curves in Γ is greater than $3g - 2$, then there exists a pair of curves in Γ that bounds an annulus in Σ .

Proof. Let $\Gamma := \{\gamma_1, \dots, \gamma_g, \dots, \gamma_{g+k}\}$, $k > 2g - 2$, be a collection of simple closed curves in Σ that are homotopically non-trivial and pair-wise disjoint. Without loss of generality, we may assume that every component of Σ contains an element of Γ ; in particular, we may assume that Σ has no spherical components. If each component of Σ has genus 1, namely it is a 2-torus, then since $g + k > g$, the number of curves in Γ is greater than the number of components. Therefore, at least one component contains two elements of Γ and thus the lemma holds in this case. Similar reasoning demonstrates that it suffices to prove the lemma under the additional hypothesis that none of the components of Σ that contains only one element of Γ is a 2-torus. Otherwise, we would remove such a component from Σ and such an element from the collection.

By definition of genus, $\Sigma - [\bigcup_{i=1}^{g+k} \gamma_i]$ consists of at least $k + 1$ connected components. Namely, $\Sigma - [\bigcup_{i=1}^{g+k} \gamma_i] = \Sigma_1 \cup \dots \cup \Sigma_n$ with $n \geq k + 1$. Suppose that none of these components is an annulus. Then, for each $i = 1, \dots, n$, $\chi(\Sigma_i) \leq -1$, where χ denotes the Euler characteristic function. Thus,

$$2 - 2g \leq \chi(\Sigma) = \chi\left(\bigcup_{i=1}^n \Sigma_i\right) = \sum_{i=1}^n \chi(\Sigma_i) \leq -n.$$

Therefore, $n \leq 2g - 2$ which gives that $g + k \leq g + n - 1 \leq 3g - 3$. In other words, if the number of curves in Γ is greater than $3g - 3$, then at least one component of $\Sigma - [\bigcup_{\gamma \in \Gamma} \gamma]$ is an annulus, and by our previous discussion, the boundary curves of this annulus are distinct elements in Γ . This completes the proof of the claim. \square

In light of Lemma 5.4, if a Riemann minimal example occurs, let n be sufficiently large so that the number of curves in equation (3) is greater than $3g - 2$. By Claim 5.3 such curves are homotopically non-trivial in M_n and thus, using Lemma 5.4 gives that at least two of them bound an annulus in M_n . This contradicts Claim 5.3 and finishes the proof of the proposition. \square

6 Global properties due to singular points

In this section we study what implications the presence of points in the singular set of convergence Δ for the sequence M_n has for the global geometry of the surfaces M_n nearby a fixed point in Δ .

Let $q \in \Delta$ be a singular point. By the previous section (see the discussion after equation (2)) and after replacing by a subsequence, there exists a sequence of points p'_n converging to q and sequences of numbers δ_n, ρ_n converging to zero, with $\lim_{n \rightarrow \infty} \frac{\rho_n}{\delta_n} = \infty$, such that $\widetilde{M}_n = \frac{1}{\delta_n} [\mathbb{B}(\rho_n) \cap M_n]$ converges (with multiplicity one or two) on compact subsets of \mathbb{R}^3 to either a catenoid \mathcal{C} or a properly embedded minimal surface with positive genus. When only the former happens, we say that q is a *catenoid singular*

point. Note that since the genus is additive and the genus of M_n is bounded by g independently of n , after replacing the sequence M_n by a subsequence, the number of singular points that are not catenoid singular points is at most g . Indeed we will show later in Proposition 9.2 that all singular points are catenoid singular points.

Throughout the rest of the section we will deal with catenoid singular points and will assume that there are at most g singular points that are not of catenoid-type. Let $q \in \Delta$ be a catenoid singular point and let \mathcal{C} denote a limit catenoid related to q with related points p'_n converging to q as described in the previous paragraph. Let $l_{\mathcal{C}}$ denote the line in \mathbb{R}^3 such that \mathcal{C} is rotationally invariant around that line, and let $\Pi_{\mathcal{C}}$ denote the plane perpendicular to $l_{\mathcal{C}}$ that is a plane of symmetry for \mathcal{C} . Let $\gamma_{\mathcal{C}}$ denote the geodesic circle in \mathcal{C} that is obtained by intersecting $\Pi_{\mathcal{C}}$ with \mathcal{C} and let $c_{\mathcal{C}}$ denote the point that is the center of this circle. We can also associate a sequence of simple closed curves $\gamma_n(q) \subset M_n$ corresponding to the curves in $\widetilde{M}_n \cap \Pi_{\mathcal{C}}$ that are converging to $\gamma_{\mathcal{C}}$ (if the multiplicity of convergence is two then we make a choice for one of the two almost-circles in $\widetilde{M}_n \cap \Pi_{\mathcal{C}}$ that give rise to the limit \mathcal{C}). We call the curve $\gamma_n(q)$ a *singular loop at q* and we denote the points in N corresponding to $c_{\mathcal{C}}$ by $c_n(q)$ and we refer to $c_n(q)$ as the *center of $\gamma_n(q)$* .

Recall that given a catenoid \mathcal{C} , the flux of $\gamma_{\mathcal{C}}$ is non-zero. Therefore, by the nature of the convergence, the flux of $\gamma_n(q)$ is also non-zero. Since the flux is a homological invariant, this implies that $\gamma_n(q)$ is homotopically non-trivial in M_n .

Suppose $q_1, q_2 \in \Delta$ are two distinct catenoid singular points that are the limit of respective points $p'_n(1), p'_n(2) \in M_n$, as described in the previous paragraphs, and let $\gamma_n(q_1), \gamma_n(q_2)$ be sequences of singular loops at q_1 and q_2 . Suppose that $\gamma_n(q_1) \cup \gamma_n(q_2)$ is the boundary of an annulus $\widetilde{A}_n(q_1, q_2)$ in M_n . Since the lengths of $\gamma_n(q_1)$ and $\gamma_n(q_2)$ are going to zero, the singular loops lift to closed curves in \mathbb{R}^3 and the annulus $\widetilde{A}_n(q_1, q_2)$ lifts to an annulus $A_n(q_1, q_2)$ in \mathbb{R}^3 . Abusing the notation, we denote the boundary of $A_n(q_1, q_2)$ by $\gamma_n(q_1) \cup \gamma_n(q_2)$ and let $c_n(q_1)$ and $c_n(q_2)$ be points in \mathbb{R}^3 corresponding to the centers of $\gamma_n(q_1)$ and $\gamma_n(q_2)$ and related to the chosen lifts. Let $l_n(q_1, q_2)$ be the straight line containing $c_n(q_1)$ and $c_n(q_2)$ and let $C_n(q_1, q_2, R)$ denote the cylinder of radius R around $l_n(q_1, q_2)$. Thus, by construction and for n sufficiently large, $\gamma_n(q_1) \cup \gamma_n(q_2) \subset C_n(q_1, q_2, \delta_n)$ but $A_n(q_1, q_2)$ is NOT contained in $C_n(q_1, q_2, \delta_n)$. The mean curvature comparison principle shows that $A_n(q_1, q_2)$ is not contained in $C_n(q_1, q_2, \frac{1}{2H_n})$.

Let z_n be a point in $A_n(q_1, q_2)$ that is farthest away from $l_n(q_1, q_2)$. To simplify the notation, after applying a sequence of translations of \mathbb{R}^3 , assume that the origin $\vec{0} \in l_n(q_1, q_2)$ and the projection of z_n onto $l_n(q_1, q_2)$ is the origin. Let r_{z_n} be the ray $\{s \frac{z_n}{|z_n|} \mid s > 0\}$ and for $t \in (0, |z_n|]$, let $\Pi(z_n)_t$ be the plane perpendicular to r_{z_n} at the point $t \frac{z_n}{|z_n|}$. Note that the mean curvature vector at z_n is $-H_n \frac{z_n}{|z_n|}$. Abusing the notation, let $A_n(q_1, q_2)$ denote the connected component of $A_n(q_1, q_2) - \Pi(z_n)_{\delta_n}$ that contains z_n and let \mathcal{H}_n denote the half-space of \mathbb{R}^3 that contains z_n and has $\Pi(z_n)_{\delta_n}$ as its boundary. Since \mathcal{H}_n is simply connected and $\partial A_n(q_1, q_2)$ is contained in $\Pi(z_n)_{\delta_n}$, $A_n(q_1, q_2)$ separates \mathcal{H}_n into two components. One of these two components is bounded and we denote its closure by G_{A_n} . Since the mean curvature vector at z_n is $-H_n \frac{z_n}{|z_n|}$

and z_n is a point in $A_n(q_1, q_2)$ that is furthest away from $\Pi(z_n)_{\delta_n}$, then G_{A_n} is mean convex. Recall that by Theorem 3.1 and its proof, the component M'_n of $\Pi^{-1}(M_n)$ in \mathbb{R}^3 that contains $A_n(q_1, q_2)$ separates \mathbb{R}^3 into two components. One of these components is mean convex and we denote its closure by $G_{M'_n}$. Let $W_n = G_{A_n} \cap G_{M'_n}$. Note that $z_n \in \partial W_n$ is a point of W_n that is furthest away from $\Pi(z_n)_{\delta_n}$.

A standard application of the Alexandrov reflection principle to the compact mean-convex region W_n , using the family of planes $\Pi(z_n)_t$, gives that the connected component $A_n^+(q_1, q_2)$ of $A_n(q_1, q_2) - \Pi(z_n)_{\frac{\delta_n + |z_n|}{2}}$ containing z_n is graphical over its projection to $\Pi(z_n)_{\frac{\delta_n + |z_n|}{2}}$ and the reflected image $A_n^-(q_1, q_2)$ of $A_n^+(q_1, q_2)$ in the plane $\Pi(z_n)_{\frac{\delta_n + |z_n|}{2}}$ intersects M'_n only along the boundary of $A_n^+(q_1, q_2)$. Since δ_n is going to zero as n goes to infinity, and $|z_n| \geq \frac{1}{2H_n} \geq \frac{1}{2b}$, then we can assume that

$$|z_n| - \frac{1}{6b} > \frac{\delta_n + |z_n|}{2}, \quad (4)$$

that is the distance from z_n to the plane $\Pi(z_n)_{\frac{\delta_n + |z_n|}{2}}$ is at least $\frac{1}{6b}$.

Lemma 6.1 *Let $F_n(q_1, q_2)$ be the region of \mathbb{R}^3 with boundary $A_n^+(q_1, q_2) \cup A_n^-(q_1, q_2)$. Then,*

1. $F_n(q_1, q_2)$ is a subdomain of $G_{M'_n}$;
2. the universal covering map $\Pi: \mathbb{R}^3 \rightarrow N$ is injective on $F_n(q_1, q_2)$.

Proof. The first item in the lemma is clear since $A_n^-(q_1, q_2)$ is contained in $G_{M'_n}$.

To prove item 2, we need to show that any non-trivial covering transformation (translation) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has the property

$$T(F_n(q_1, q_2)) \cap F_n(q_1, q_2) = \emptyset.$$

By separation properties, it suffices to show that T has the property

$$T(A_n^+(q_1, q_2) \cup A_n^-(q_1, q_2)) \cap [A_n^+(q_1, q_2) \cup A_n^-(q_1, q_2)] = \emptyset.$$

Arguing by contradiction, suppose that

$$T(A_n^+(q_1, q_2) \cup A_n^-(q_1, q_2)) \cap [A_n^+(q_1, q_2) \cup A_n^-(q_1, q_2)] \neq \emptyset.$$

It then follows from Theorem 3.1 and its proof that $T(G_{M'_n}) = G_{M'_n}$. Since the annulus $A_n(q_1, q_2)$ in \mathbb{R}^3 is a lift of the annulus $\tilde{A}_n(q_1, q_2)$ in N , then

$$T(A_n(q_1, q_2)) \cap A_n(q_1, q_2) = \emptyset.$$

In particular

$$T(A_n^+(q_1, q_2)) \cap A_n^+(q_1, q_2) = \emptyset.$$

Moreover,

$$T(A_n^+(q_1, q_2)) \cap A_n^-(q_1, q_2) = \emptyset$$

because $\text{Int}(A_n^-(q_1, q_2)) \subset \text{Int}(G_{M'_n})$ and $T(\text{Int}(G_{M'_n})) = \text{Int}(G_{M'_n})$. Similarly, we have that $T(A_n^-(q_1, q_2)) \cap A_n^+(q_1, q_2) = \emptyset$. Thus, it remains to show that

$$T(A_n^-(q_1, q_2)) \cap A_n^-(q_1, q_2) = \emptyset.$$

This follows from the already proved set equation $T(A_n^+(q_1, q_2)) \cap A_n^-(q_1, q_2) = \emptyset$ and because $T(A_n^-(q_1, q_2))$ is the reflection of $T(A_n^+(q_1, q_2))$ across the plane $T(\Pi(z_n)_{|z_n| - \frac{\delta_n + |z_n|}{2}})$. \square

Let $A_n^*(q_1, q_2)$ be the connected component of $A_n^+(q_1, q_2) - \Pi(z_n)_{|z_n| - \frac{1}{12b}}$ that contains z_n . By construction, a point in $A_n^*(q_1, q_2)$ is a point in $A_n^+(q_1, q_2)$ at distance at least $\frac{1}{12b}$ from the boundary of $A_n^+(q_1, q_2)$. Thus, applying the uniform curvature estimates in [26] for oriented graphs with constant mean curvature (graphs are stable with curvature estimates away from their boundaries, which do not depend on the value of the constant mean curvature), gives that points in $A_n^*(q_1, q_2)$ satisfy a uniform curvature estimate depending only on b .

Furthermore, the same standard application of the Alexandrov reflection principle implies the following. Let $G(q_1, q_2) \subset G_{M'_n}$ be the bounded open region of \mathbb{R}^3 contained between $A_n^*(q_1, q_2)$ and its reflection across the plane $\Pi(z_n)_{|z_n| - \frac{1}{12b}}$. Then $G(q_1, q_2)$ is contained in the interior of W_n .

Lemma 6.2 *Let $\tilde{G}(q_1, q_2)$ denote the image of $G(q_1, q_2)$ in N via the universal covering map. There exists $\varepsilon_1 > 0$ such that $\text{Volume}(\tilde{G}(q_1, q_2)) > \varepsilon_1$, and ε_1 is independent of the lower bound for the mean curvature of M_n , that is a , and of n for n sufficiently large. Moreover, if $\tilde{G}(q_1, q_2)$ and $\tilde{G}(p_1, p_2)$ are regions of N related to two disjoint annuli, then these regions are disjoint.*

Proof. By Lemma 6.1, it suffices to prove that there exists $\varepsilon_1 > 0$ such that

$$\text{Volume}(G(q_1, q_2)) > \varepsilon_1,$$

independently of n for n sufficiently large and such that ε_1 only depends on b .

The connected component $A_n^*(q_1, q_2)$ defined above is a graph over $\Pi(z_n)_{|z_n| - \frac{1}{12b}}$ and the distance from $z_n \in A_n^*(q_1, q_2)$ to the plane $\Pi(z_n)_{|z_n| - \frac{1}{12b}}$ is at least $\frac{1}{12b}$ by (4). Therefore, by curvature estimates for constant mean curvature graphs, $A_n^*(q_1, q_2)$ contains a subgraph G_{sub} with $z_n \in G_{sub}$ satisfying the following two properties.

- The projection of G_{sub} to $\Pi(z_n)_{|z_n| - \frac{1}{12b}}$ contains a disk of radius ρ bounded from below and this bound depends only on curvature estimates for constant mean curvature graphs.
- The distance from G_{sub} to the plane $\Pi(z_n)_{|z_n| - \frac{1}{12b}}$ is at least $\frac{1}{24b}$.

Therefore, a truncated cylinder of radius ρ and height $\frac{1}{12b}$ is contained in $G(q_1, q_2)$ which gives that $\text{Volume}(G(q_1, q_2)) > \frac{\pi\rho^2}{12b}$.

The fact that if $\tilde{G}(q_1, q_2)$ and $\tilde{G}(p_1, p_2)$ are regions of N related to two disjoint annuli, then these regions are disjoint, follows from Lemma 6.1 and its proof. \square

7 Bounding the number of singular points independently of the lower positive bound a of the mean curvature

In this section we bound the number of points in Δ . Since the number of singular points that are not catenoid singular points is at most g , it suffices to bound the number of catenoid singular points.

Let $\{q_1, \dots, q_m\} \in \Delta$ be a collection of catenoid singular points. It is important to remark that in what follows, the integer $n \in \mathbb{N}$ is chosen sufficiently large so that the estimates of previous sections, such as those in appearing in (4), make sense at each of the points in $\{q_1, \dots, q_m\}$. By definition and the discussion in the previous section, to each q_i corresponds a sequence of singular loops $\gamma_n(q_i)$ and such loops are homotopically non-trivial. Thus, by applying Claim 5.4, if $m > k(3g - 2)$ we obtain at least k annuli A_1, \dots, A_k with pairs of singular loops as their boundaries. Note that if $A_i \cap A_j \neq \emptyset$ then their intersection must be an annulus with a pair of singular loops as its boundary. Therefore, after possibly replacing the collection of annuli A_1, \dots, A_k with a different collection, we can assume that the annuli are pairwise disjoint.

For every $i = 1, \dots, k$, let $\tilde{G}_i \subset G_{M_n}$ denote the region of N , as described in Lemma 6.2 of the previous section. By that lemma, there exists $\varepsilon_1 > 0$ depending only on b (i.e., not depending on a) and independent of n and i , such that $\text{Volume}(\tilde{G}_i) \geq \varepsilon_1$ and that $\tilde{G}_i \cap \tilde{G}_j = \emptyset$ if $i \neq j$. Then we have the following inequality:

$$k\varepsilon_1 \leq \sum_{i=1}^k \text{Volume}(\tilde{G}_i) = \text{Volume}\left(\bigcup_{i=1}^k \tilde{G}_i\right) \leq \text{Volume}(N).$$

Therefore,

$$k \leq \frac{\text{Volume}(N)}{\varepsilon_1}$$

which implies that

$$m \leq \frac{\text{Volume}(N)}{\varepsilon_1}(3g - 2).$$

Thus, adding also the bound for the number of singular points that are not catenoid singular points, the previous inequality gives that the number of singular points is bounded by

$$\frac{\text{Volume}(N)}{\varepsilon_1}(3g - 2) + g.$$

Remark 7.1 Note that the proof that the number of singular points is bounded does NOT use the fact that the area of M_n is becoming arbitrarily large. Also, this bound on the number of singular points does not depend on the lower bound a for the constant mean curvature of M_n .

8 The final contradiction

In this section we prove that the area of M_n is uniformly bounded from above. This contradicts the fact that $\text{Area}(M_n) > n$ and this contradiction will finish the proof of Theorem 4.1.

Let $\Delta := \{q_1, \dots, q_m\}$ be the set of singular points. The results in the previous section give that

$$m \leq \frac{\text{Volume}(N)}{\varepsilon_1} (3g - 2) + g.$$

Note that since M_n separates N and the norms of the second fundamental forms of M_n are uniformly bounded on compact sets of $N - \Delta$, by applying Proposition 2.6 the following holds. If $p \in N - \Delta$ and $\varepsilon > 0$ is such that $B_N(p, \varepsilon) \cap \Delta = \emptyset$, then there exists a constant $T(\varepsilon)$ such that $\text{Area}(M_n \cap B_N(p, \frac{\varepsilon}{2})) < T(\varepsilon)$. A standard compactness argument then gives that there exists a surface M_∞ properly immersed in $N - \Delta$ such that, up to a subsequence, $M_n - \Delta$ converges to M_∞ on compact subsets of $N - \Delta$. The surface M_∞ has constant mean curvature H , for a certain $H \in [a, b]$. Since Δ is finite, there exists $r > 0$ such that for any $q \in \Delta$, $B_N(q, r) \cap \Delta = q$ and $B_N(q, \tau) \cap M_\infty \neq \emptyset$, for any $\tau \in (0, r]$.

Claim 8.1 *The sequence $M_n - \Delta$ converges to M_∞ with multiplicity one and M_∞ is strongly Alexandrov embedded in $N - \Delta$.*

Proof. Recall that M_n separates N into two regions and one of them, denoted by G_{M_n} , is mean convex. Given $p \in M_\infty$, then there exists $\varepsilon > 0$ such that a pointed connected component of $B_N(p, \varepsilon) \cap M_\infty$, which we denote by $\Omega(p)$, is a graph over the tangent plane to M_∞ at p , $T_p M_\infty$, and it is the limit of a sequence of graph $U_n(p) \subset M_n$ over $T_p M_\infty$. Note that $\Omega(p)$ has a well-defined mean curvature vector obtained as the limit of the mean curvature vectors of $U_n(p)$. If M_n contained more than one graph over $T_p M_\infty$ converging to $\Omega(p)$, since M_n separates N , then the mean curvature vectors would change orientation on consecutive graphs in M_n and the mean curvature vector on $\Omega(p)$ would not be well-defined. This proves that $M_n - \Delta$ converges to M_∞ with multiplicity one.

By the previous argument and the fact that the surfaces $M_n - \Delta$ converge to M_∞ with multiplicity one, then the connected open regions that are $\text{Int}(G_{M_n}) - \Delta$ converge to an open region W in $N - \Delta$ and $\partial W = M_\infty$. This shows that M_∞ is strongly Alexandrov embedded in $N - \Delta$, which finishes the proof of the claim. \square

By the nature of the convergence with multiplicity one, and since M_∞ is properly immersed in $N - \Delta$, for any $\varepsilon > 0$, there exists $K(\varepsilon)$ such that

$$\lim_{n \rightarrow \infty} \text{Area}(M_n \cap [N - \bigcup_{i=1}^m B_N(q_i, \varepsilon)]) = \text{Area}(M_\infty \cap [N - \bigcup_{i=1}^m B_N(q_i, \varepsilon)]) < K(\varepsilon).$$

Fix $\varepsilon > 0$ such that $4e^{-\varepsilon b} \geq 2$, where b is the upper bound for the mean curvature of M_n , $B_N(q_i, 2\varepsilon)$ is an open ball in N and for any $i, j \in \{1, \dots, m\}$ with $i \neq j$, then $B_N(q_i, 2\varepsilon) \cap B_N(q_j, 2\varepsilon) = \emptyset$. Then, by the previous argument, for each $i := 1, \dots, m$ and n sufficiently large,

$$\text{Area}(M_n \cap [N - \bigcup_{i=1}^m B_N(q_i, \varepsilon)]) < K(\varepsilon) + 1.$$

Recall that $H_n \leq b$. By the monotonicity formula for H_n -surfaces, see for instance [29], it follows that

$$\frac{\text{Area}(M_n \cap B_N(q_i, 2\varepsilon))}{4\varepsilon^2} \geq e^{-\varepsilon H_n} \frac{\text{Area}(M_n \cap B_N(q_i, \varepsilon))}{\varepsilon^2} \geq e^{-\varepsilon b} \frac{\text{Area}(M_n \cap B_N(q_i, \varepsilon))}{\varepsilon^2}$$

This implies that

$$\text{Area}(M_n \cap B_N(q_i, 2\varepsilon)) \geq 4e^{-\varepsilon b} \text{Area}(M_n \cap B_N(q_i, \varepsilon)) \geq 2\text{Area}(M_n \cap B_N(q_i, \varepsilon)).$$

Therefore, for n sufficiently large,

$$\begin{aligned} \text{Area}(M_n \cap B_N(q_i, \varepsilon)) &\leq \text{Area}(M_n \cap B_N(q_i, 2\varepsilon)) - \text{Area}(M_n \cap B_N(q_i, \varepsilon)) \\ &= \text{Area}(M_n \cap [B_N(q_i, 2\varepsilon) - B_N(q_i, \varepsilon)]) \\ &< K(\varepsilon) + 1. \end{aligned}$$

Finally, this implies that for n large,

$$\begin{aligned} \text{Area}(M_n) &= \text{Area}(M_n \cap [N - \bigcup_{i=1}^m B_N(q_i, \varepsilon)]) + \sum_{i=1}^m \text{Area}(M_n \cap B_N(q_i, \varepsilon)) \\ &< (m+1)(K(\varepsilon) + 1). \end{aligned}$$

Since ε is fixed, independent of n , and m is at most $\frac{\text{Volume}(N)}{\varepsilon_1}(3g-2)+g$, this contradicts the fact $\text{Area}(M_n) > n$. This contradiction completes the proof of Theorem 4.1. \square

8.1 The proof of Theorem 4.1 for disconnected H -surfaces M

We next explain why Theorem 4.1 holds for another choice of constant $A'(N, a, b, g)$ when M in its statement is not necessarily connected. First we may assume that the constants $A(N, a, b, g)$ given in Theorem 4.1 are increasing as a function of the genus g . Also notice that any flat 3-torus T with injectivity radius I_0 and diameter d has a fixed upper bound $V(I_0, d)$ on its volume. Observe that any collection of pairwise disjoint embedded H -spheres in such a T with $H \in [a, b]$ bound a family of pairwise disjoint balls in T with volume at least $\frac{4}{3}\pi(\frac{1}{b})^3$ and the sum of these volumes is bounded by $V(I_0, d)$. Thus, the number of spherical components of a disconnected closed H -surface Σ of genus g in T is bounded by some constant $S(I_0, d, b) \in \mathbb{N}$. Therefore if M is a possibly *disconnected* surface satisfying the other hypotheses of Theorem 4.1, then it can have at most $S(I_0, d, b) + g$ components. Hence, the area of M is at most

$$A'(N, a, b, g) = [S(I_0, d, b) + g]A(N, a, b, g),$$

which proves our desired claim.

9 Compactness of H -surfaces in a flat 3-torus

In this section, we prove Theorem 1.2 from the Introduction. Let N be a flat 3-torus and let M_n be a sequence of closed H_n -surfaces in N , $H_n \in [a, b]$, of genus at most g . Theorem 4.1 and the discussion at the end of the previous section gives that there exists a constant C independent of n such that

$$\sup_n \text{Area}(M_n) < C.$$

By the results in Section 7 and in particular note Remark 7.1, after passing to a subsequence, there exists a possibly empty finite set of points Δ , namely the set of singular points of convergence, such that M_n has locally bounded norm of the second fundamental form in $N - \Delta$.

The standard compactness argument already used in Section 8, see Claim 8.1, gives that there exists a surface M_∞ strongly Alexandrov embedded in $N - \Delta$ such that, up to a subsequence, $M_n - \Delta$ converges to M_∞ on compact subsets of $N - \Delta$ with multiplicity one. The surface M_∞ has constant mean curvature H , for a certain $H \in [a, b]$. Moreover the convergence to M_∞ has multiplicity one which implies that the genus of M_∞ is at most g . Recall that Δ is in the closure of M_∞ .

Claim 9.1 *The points in Δ are removable singularities for M_∞ .*

Proof. By Theorem 4.1 and the discussion at the end of the previous section, there exists a constant $C > 0$ such that $\sup_n \text{Area}(M_n) < C$. Gauss-Bonnet Theorem together with the Gauss equation gives that

$$2\pi\chi(M_n) = \int_{M_n} K_{M_n} = \int_{M_n} (2H_n^2 - \frac{|A_{M_n}|^2}{2}).$$

Since the genus of M_n is at most g , $H_n \leq b$ and $\text{Area}(M_n) < C$ then, using the previous inequality gives that there exists a constant C_a , independent of n , such that

$$\int_{M_n} |A_{M_n}|^2 < C_a. \quad (5)$$

Since $\int_{M_n} |A_{M_n}|^2 = \int_{M_n - \Delta} |A_{M_n}|^2$, by the nature of the convergence with multiplicity one it follows that

$$\int_{M_\infty} |A_{M_\infty}|^2 \leq C_a.$$

By applying a rescaling argument around each point $q \in \Delta$, this gives that there exists a constant $C_b > 0$ such that for any $p \in M_\infty$,

$$|A_{M_\infty}|(p) < \frac{C_b}{d_N(p, \Delta)}.$$

Since M_∞ is a weak H -lamination of $N - \Delta$, Theorem 1.2 in [12] implies that M_∞ extends smoothly across Δ to a weak H -lamination of N and the points in Δ are removable singularities for M_∞ . \square

Since M_∞ extends across Δ , if by abusing the notation we denote by M_∞ the related surface $M_\infty \cup \Delta$, then M_∞ is strongly Alexandrov embedded in N .

It remains to show that the singular set of convergence Δ is contained in the set of points of self-intersection of M_∞ . Note that if $p \in M_\infty$ is not a point of self-intersection of M_∞ , then

$$\lim_{r \rightarrow 0} \frac{\text{Area}(M_\infty \cap [B_N(p, 2r) - B_N(p, r)])}{\pi r^2} = 3. \quad (6)$$

Instead, by the description at point of self-intersection, if $p \in M_\infty$ is a point of self-intersection of M_∞ , then

$$\lim_{r \rightarrow 0} \frac{\text{Area}(M_\infty \cap [B_N(p, 2r) - B_N(p, r)])}{\pi r^2} = 6. \quad (7)$$

Recall that by the monotonicity formula for H_n -surfaces, since $H_n \leq b$, it follows that for any $q \in M_n$ and $r_1 < 2r_2 < I_0$ the following holds,

$$\begin{aligned} \frac{\text{Area}(M_n \cap B_N(q, r_2))}{\pi r_2^2} &\geq e^{-(r_2 - r_1)H_n} \frac{\text{Area}(M_n \cap B_N(q, r_1))}{\pi r_1^2} \\ &\geq e^{-r_2 b} \frac{\text{Area}(M_n \cap B_N(q, r_1))}{\pi r_1^2}. \end{aligned}$$

Therefore, applying this inequality, we obtain that

$$\begin{aligned} \frac{\text{Area}(M_n \cap [B_N(q, 2r_2) - B_N(q, r_2)])}{\pi r_2^2} &\geq (4e^{-r_2 b} - 1) \frac{\text{Area}(M_n \cap B_N(q, r_2))}{\pi r_2^2} \\ &\geq (4e^{-r_2 b} - 1) e^{-r_2 b} \frac{\text{Area}(M_n \cap B_N(q, r_1))}{\pi r_1^2} \quad (8) \\ &\geq \frac{8}{3} \frac{\text{Area}(M_n \cap B_N(q, r_1))}{\pi r_1^2}, \end{aligned}$$

if r_2 is chosen so that $4e^{-r_2 b} - 1 \geq \frac{8}{3}$.

Let $q \in \Delta$ be a singular point. If q is a catenoid singular point then there exists $q_n \in M_n$ and $\delta_n > 0$ such that $\lim_{n \rightarrow \infty} q_n = q$, $\lim_{n \rightarrow \infty} \delta_n = 0$ and

$$\frac{\text{Area}(M_n \cap B_N(q_n, \delta_n))}{\pi \delta_n^2} \geq \frac{3}{2}. \quad (9)$$

If $q \in \Delta$ is a singular point that is NOT a catenoid singular point then the limit surface given by Proposition 5.1 is a properly embedded minimal surface with positive genus at most g ; furthermore it has finite total curvature by equation (5). Let L_∞ denote this limit surface. Then, by the main theorem in [27], L_∞ has at least three ends and thus there exists $q_n \in M_n$ and $\delta_n > 0$ such that $\lim_{n \rightarrow \infty} q_n = q$, $\lim_{n \rightarrow \infty} \delta_n = 0$ and

$$\frac{\text{Area}(M_n \cap B_N(q_n, \delta_n))}{\pi \delta_n^2} \geq \frac{5}{2}. \quad (10)$$

If $q \in \Delta$ is NOT a point of self-intersection for M_∞ , by equation (6) we can fix $r > 0$ arbitrarily small such that

$$\frac{\text{Area}(M_\infty \cap [B_N(q, 2r) - B_N(q, r)])}{\pi r^2} < \frac{7}{2}.$$

However, by equations (8), (9) and (10),

$$\frac{\text{Area}(M_n \cap [B_N(q_n, 2r) - B_N(q_n, r)])}{\pi r^2} \geq 4 > \frac{7}{2}.$$

Since the convergence away from the singular points is smooth with multiplicity one, it holds that

$$\lim_{n \rightarrow \infty} \frac{\text{Area}(M_n \cap [B_N(q_n, 2r) - B_N(q_n, r)])}{\pi r^2} = \frac{\text{Area}(M_\infty \cap [B_N(q, 2r) - B_N(q, r)])}{\pi r^2}.$$

When n is sufficiently large, this leads to a contradiction. This finishes the proof that Δ is contained in the set of points of self-intersection of M_∞ , which finishes the proof of the theorem.

The previous argument can be used to rule out the occurrence of singular points that are not catenoid singular points.

Proposition 9.2 *Any singular point in Δ is a catenoid singular point.*

Proof. Suppose $q \in \Delta$ is a singular point that is not a catenoid singular point. By equation (6) and equation (7) we can fix $r > 0$ arbitrarily small such that

$$\frac{\text{Area}(M_\infty \cap [B_N(q, 2r) - B_N(q, r)])}{\pi r^2} < \frac{13}{2}.$$

However, by equation (8) and (10),

$$\frac{\text{Area}(M_n \cap [B_N(q_n, 2r) - B_N(q_n, r)])}{\pi r^2} \geq \frac{8}{3} \cdot \frac{5}{2} > \frac{13}{2},$$

where $\lim_{n \rightarrow \infty} q_n = q$. Since the convergence away from the singular points is smooth with multiplicity one, this leads to a contradiction, for n sufficiently large. \square

10 Analysis of the area bound and the proofs of Theorems 1.1 and 1.2

In this section we prove that the area bound only depends on certain properties of the flat 3-torus. Throughout this section we let $\mathcal{T}(d, I_0)$ be the space of flat 3-tori, satisfying

1. d is an upper bound on the diameter of N ;
2. $I_0 > 0$ is a lower bound on the injectivity radius of N .

Recall that the first property gives that the volume of each flat 3-torus in $\mathcal{T}(d, I_0)$ is bounded from above by $\frac{\pi d^3}{6}$. Note also that we can view $\mathcal{T}(d, I_0)$ as a compact set of Riemannian metrics on the smooth manifold $T = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Since the universal covers of each flat 3-torus in $\mathcal{T}(d, I_0)$ are all \mathbb{R}^3 with the flat metric, we can view these flat 3-tori to be quotients of \mathbb{R}^3 by smoothly varying lattices.

We now prove Theorem 1.1. Arguing by contradiction, suppose that $f_n: M_n \rightarrow N_n$ is a sequence of closed, possibly disconnected, H -surfaces where:

1. $N_n \in \mathcal{T}(d, I_0)$;
2. the area of M_n is greater than n ;
3. the genus of M_n is at most some fixed $g \in \mathbb{N}$.

Suppose that the flat 3-tori N_n converge to a flat 3-torus N , i.e., the metrics converge to a flat metric on T . Then we can view the injective mappings $f_n: M_n \rightarrow N_n$ to correspond to quasi-isometric mappings into N .

There are two cases to consider. If the injectivity radii of the surfaces M_n are bounded away from zero, then the norms of their second fundamental forms are bounded and the argument in the proof of Claim 4.2 gives a contradiction; more precisely, the surfaces M_n have uniform regular ε -neighborhoods on their mean convex sides that, after replacing by a subsequence, converge smoothly with multiplicity one to a regular ε -neighborhood on the mean convex side of a smooth strongly Alexandrov embedded closed surface of genus at most g and such convergence has multiplicity one.

Suppose now that, after replacing by a subsequence, the injectivity radius of M_n is less than $1/n$ and that, after choosing a subsequence, there is a point $q_1 \in N$ and points $p_n \in f_n(M_n) \subset N$ where the $I_{M_n}(p_n) < 1/n$; here we are viewing $f_n(M_n)$ as being contained in both N_n and the related limit N . Arguing exactly as in the proof of Theorem 4.1, we can define in a new subsequence (also labeled as M_n) and a set $\Delta \subset N$ which is the set of singular points of convergence for M_n . As previously, rescaling arguments on the scale of the injectivity radius, show that for any $q \in \Delta$, we can find points $p'_n \in M_n$ of almost minimal injectivity radius such that in small balls in N centered at the points p'_n , the surfaces M_n have the appearance of a complete properly embedded minimal surfaces M_∞ in \mathbb{R}^3 with finite genus at most g or a parking garage structure of \mathbb{R}^3 with two oppositely handed columns.

As in our previous study, after replacing by a subsequence, we may assume that at most g points in Δ can produce a limit minimal surface M_∞ of positive genus. As before, the only possible limit minimal surface M_∞ of genus zero is the catenoid. From this point on, all of the arguments that go into the proof of Theorem 4.1 work to show that the set Δ is finite and all of these points correspond to the case that the limit surface M_∞ that forms near them is of catenoid type; as before, these arguments also yield a contradiction to the assumption that the areas of the originally chosen surfaces M_n is unbounded. This contradiction completes the proof of Theorem 1.1.

Remark 10.1 The arguments in this section can be applied to prove that Theorem 1.2 holds in a more general setting. Namely when the surfaces M_n have constant mean curvature $H_n \in [a, b]$, $a, b \in (0, \infty)$, and lie in flat 3-tori N_n whose injective radii are at least $I_0 > 0$ and whose diameters are bounded from above by some $d > 0$, then a subsequence of the 3-tori converges to a flat 3-torus N_∞ and a subsequence of the related surfaces converges to a strongly Alexandrov embedded surface M_∞ in N_∞ .

11 Some images of genus 3 H-surfaces in flat 3-tori and the dependence of $A(g, a, b, d, I_0)$ on its variables

One way to construct examples of triply periodic surfaces of non-zero constant mean curvature 1 in \mathbb{R}^3 is to solve Plateau's problem for a geodesic polygon in the sphere $\mathbb{S}^3 = \{|x| = 1 \mid x \in \mathbb{R}^4\}$, isometrically map these least area surfaces into \mathbb{R}^3 by the Lawson correspondence [7], and then extend them to all of \mathbb{R}^3 by reflections and translations. In Figure 4 we present several images of the fundamental regions of such constant mean curvature surfaces in a fundamental region of the flat 3-torus $\mathbb{R}^3/\mathbb{Z}^3$.

Remark 11.1 By way of examples, it can be shown that the choice of the constant $A(g, a, b, d, I_0)$ in the statement of Theorem 1.1 must depend on the variables g, a, b, d , once I_0 is given. We now indicate without proof what these examples are.

1. **Dependence on an upper bound of H :** Every flat 3-torus T admits for any n a collection of pairwise disjoint geodesic spheres of fixed radius and with total area greater than n ; the constant mean curvatures of these spheres necessarily goes to infinity as n tends to infinity. Also note that there exist connected examples that are flat geodesic cylinders around long closed geodesics in T and that have arbitrarily large area.
2. **Dependence on a positive lower bound of H :** If H is not bounded from below, then the flat 3-torus $T = \mathbb{R}^3/\mathbb{Z}^3$ admits H_n -surfaces M_n of genus 3 with areas greater than n and with $H_n \in (0, 1/n)$; these surfaces can be seen as small deformations of genus-3 minimal surfaces in T with area greater than n and geometrically M_n have the appearance of being an “almost totally geodesic” 2-torus in T and with two attached small “almost-catenoids,” where one of these “almost-catenoids” is placed

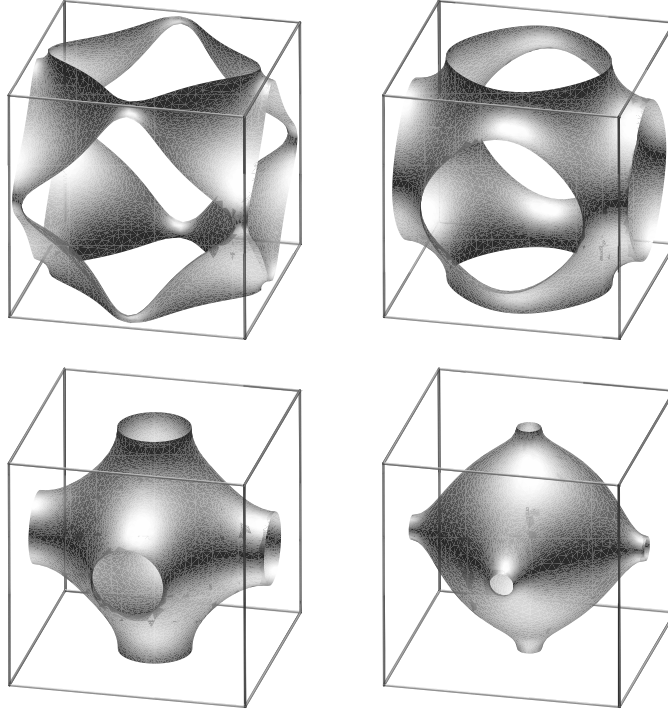


Figure 4: Presented above are 4 examples of genus-3 triply periodic constant mean curvature surfaces in a fundamental region of the 3-torus $\mathbb{R}^3/\mathbb{Z}^3$. The two surfaces in the top row have mean curvature vectors pointing away from the center of the box. The other two surfaces have mean curvature vectors pointing towards the center of the box. These images have been kindly provided by Karsten Große-Brauckmann.

at the origin $\overline{(0,0,0)} \in T$ and the other one is placed near the half-lattice point $\overline{(1/2, 1/2, 1/2)} \in T$.

3. **Dependence on an upper bound for the diameter d :** The 3-torus $T_n = \mathbb{R}^3/(\mathbb{Z} \times \mathbb{Z} \times n\mathbb{Z})$ contains intrinsically flat vertical “cylinders” (flat 2-tori) C_n of “radius $1/3$ ” and height n with area $2n\pi/3$.
4. **Dependence on an upper bound for g :** In [15] we construct disconnected closed surfaces M_n of constant mean curvature 1, genus greater than n , and area greater than n that are properly embedded in some flat 3-torus T_n and such that the sequence of 3-tori T_n converges to the flat 3-torus $T = \mathbb{R}^3/\mathbb{Z}^3$ as n tends to infinity. We hope to prove that the surfaces M_n can be chosen to be connected.

12 Outstanding Problems

The following outstanding problems are closely related to Theorems 1.1 and 1.2; these problems also provided our original motivations for the results in this paper. It follows from the proofs of Theorems 1.1 and 1.2 that if $M_n \subset N$ is a sequence of H_n -surfaces satisfying the hypotheses in Theorem 1.2 that converges to the limit surface M_∞ given in its conclusion, then: *Let $q \in N$ be a singular point of convergence of the M_n to M_∞ . Then for any $\varepsilon > 0$ sufficiently small, there exists an $N_0 = N_0(\varepsilon)$ such that for $n \geq N_0$, $\Sigma_n = \overline{B}_N(q, \varepsilon) \cap M_n$ is a connected compact surface with two boundary components.*

Conjecture 12.1 (Genus-zero Singular Points of Convergence Conjecture)

For $\varepsilon > 0$ sufficiently small and n sufficiently large, the compact surface $\Sigma_n = \overline{B}_N(q, \varepsilon) \cap M_n$ is annulus of total absolute Gaussian curvature $C(\Sigma_n) \in (4\pi - \varepsilon, 4\pi + \varepsilon)$.

The next conjecture is motivated by the compactness result Theorem 1.2. In contrast to this conjecture, recall that Traizet [31] proved that for any positive integer $g \neq 2$ and $n \in \mathbb{N}$, every flat 3-torus contains an embedded, connected closed minimal surface of genus g with area greater than n .

Conjecture 12.2 (Finiteness Conjecture) *For any $H > 0$ and $g \in \mathbb{N} \cup \{0\}$, the moduli space of non-congruent, connected closed H -surfaces of at most genus g in a fixed flat 3-torus is finite.*

Definition 12.3 A complete H -surface in a Riemannian 3-manifold X is said to have *locally finite genus* if for every point $p \in X$, there exists an $\varepsilon_p > 0$ such that the genus of $M \cap B_X(p, \varepsilon_p)$ is bounded. If for some $\varepsilon > 0$, the upper bound U on the genus of $M \cap B_X(p, \varepsilon)$ is independent of the point p , then we say that M has *ε -uniformly bounded genus with bound U* .

Conjecture 12.4 (Embedded Calabi-Yau Problem for Locally Finite Genus)

Let M be a complete 1-surface in \mathbb{R}^3 .

1. *M is proper in \mathbb{R}^3 if and only if it has locally bounded genus in \mathbb{R}^3 . Furthermore, this same properness result holds for complete, non-planar minimal surfaces embedded in \mathbb{R}^3 .*
2. *Given $\varepsilon, U > 0$, there exists $A(\varepsilon, U) > 0$ such that if M has ε -uniformly bounded genus with bound U , then, for all $p \in \mathbb{R}^3$,*

$$\text{Area}(M \cap \mathbb{B}(p, \varepsilon)) \leq A(\varepsilon, U).$$

Remark 12.5 The area estimates given in Theorem 1.1 should hold in the following more general context. Let N be a closed orientable Riemannian 3-manifold. Given positive numbers $a \leq b$ and $g \in \mathbb{N} \cup \{0\}$, there exists positive number $A(g, a, b)$ depending only on g, a, b and N such that the areas of a closed H -surfaces M with genus g and

$H \in [a, b]$ is less than $A(g, a, b)$, under the assumption that M is the oriented boundary of a subdomain of N . In particular, if N is a Riemannian homology 3-sphere, then there is an area estimate for connected, closed H -surfaces M with fixed finite genus g and $H \in [a, b]$. This generalization is work in progress in [14].

William H. Meeks, III at profmeeks@gmail.com

Mathematics Department, University of Massachusetts, Amherst, MA 01003

Giuseppe Tinaglia at giuseppe.tinaglia@kcl.ac.uk

Department of Mathematics, King's College London, London, WC2R 2LS, U.K.

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